MDS code

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Theorem 1. Given a redundancy r and a minimum distance d. An $[n, n-r, d]$-code satisfies $d \leq r+1$.

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Definition 1. A linear $[n, k, d]$ code over F with $d = n-k+1$ is called a *maximum distance separable* (MDS) code.

In other words, an MDS is a $[n, n-r, r+1]$-code.

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Theorem 2. Suppose $2 \leq r \leq q$. Let $a_1, \ldots, a_{q-1}$ be the non-zero elements of $GF(q)$. Then the matrix

$$H = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ a_1 & a_2 & \cdots & a_{q-1} & 0 & 1 & \cdots & 0 \\ a_1^2 & a_2^2 & \cdots & a_{q-1}^2 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_1^{r-1} & a_2^{r-1} & \cdots & a_{q-1}^{r-1} & 0 & & \cdots & 1 \end{bmatrix}$$

is the parity check matrix of an MDS $q+1, q+1-r, r+1$ code. Equivalently, the columns of H form a $(q+1)$-arc in $PG(r-1, q)$.

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Theorem 3. Let C be a linear $[n, k, d]$ code over a field F of q elements, where q is a prime power with a parity check matrix H. Then C has a code word of weight $w \leq l$ if and only if l columns of H are linearly dependent.

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Theorem 4. Let C be a linear $[n, k, d]$ code over F with a parity check matrix H. Then C is an MDS code if and only if every $n-k$ columns of H are linearly independent.

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Theorem 5. If a linear $[n, k, d]$ code C is MDS, then so is its dual $C^{\perp}$.

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Corollary 5[1]. Let C be an $[n, k, d]$ linear code over $F = GF(q)$. Then the following statements are equivalent.

a. C is MDS
b. Every k columns of a generator matrix G of C are linearly independent
c. Every $n-k$ columns of a parity check matrix H of C are linearly independent

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Problem 1. Show that linear $[n, 1, n]$, $[n, n-1, 2]$ and $[n, n, 1]$ codes exist over any finite field F.

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Definition 2. We call *trivial MDS codes* the $[n, 1, n]$, $[n, n-1, 2]$ and $[n, n, 1]$ codes.

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Theorem 6. The only binary MDS codes are the trivial ones.

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Definition 3. A square matrix is said to be *non-singular* if its columns are linearly independent. Given any matrix A, a $s \times s$ *square submatrix* of A is a $s \times s$ matrix consisting of the entries from some s rows and s column of A.

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Theorem 7. Let C be an $[n, k, -]$ code with parity check matrix $H = (A \quad I_{n-k})$. Then C is an MDS code if and only if every square submatrix of A is non-singular.

Proof. Let B_r be a square submatrix of A which rests upon the i_1^{th}, i_2^{th}, ..., i_r^{th} rows of A with $i_1 < i_2 < \cdots < i_r \leq n-k$. Let M_r be the square submatrix of H of order $n-k$ having the columns of A parts which occur in B_r and the ramaining $n-k-r$ columns from I_{n-k} that are not the i_1^{th},

i_2^{th}, ..., i_r^{th} columns. Thus we could always find the determinant by pivotting on the ones in the columns I_j, $j \neq i_1, i_2, \cdot, i_r$ successively. Then $\det M_r = \mathrm{p}\det B_r$. Therefore B_r is non-singular if and only if M_r is. Hence every $n-k$ columns of H are linearly independent if and only if every square submatrix of A is non-singular. ¶

Theorem 8. Let C be an $[n, k, -]$ code with generator matrix $G = (I_k \quad A)$. Then C is an MDS code if and only if every square submatrix of A is non-singular.

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Theorem 9. Let C be an $[n, k, d]$ MDS code. Then any k symbols of the code words may be taken as message symbols.

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Theorem 10. Let C be an $[n, k, d]$ code over $GF(q)$. Then C is an MDS code if and only if C has a minimum distance code word with non-zero entries in any d coordinates.

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Corollary 10[1]. The number of code words of weight $n-k+1$ in an $[n, k, d]$ MDS code over $GF(q)$ is

$$(q-1)\binom{n}{n-k+1}$$

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Problem 2. Given k and q, find the largest value, $m(k, q)$, of n such that $[n, k, n-k+1]$ MDS code exists over $GF(q)$.

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Because of Theorem 5, Problem 2 is equivalent to Problem 3.

Problem 3. Given k and q, find the largest n for which there is a $k \times n$ matrix over $GF(q)$, every k columns of which are linearly independent.

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Problem 4. Given a k-dimensional vector space V over $GF(q)$, what is the order of a largest subset of V every k vectors of which form a basis of the same?.

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Theorem 11. For any prime power q, we have $m(2, q) = q + 1$.

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Theorem 12.

$$m(k, q) = k + 1$$

for $q \leq k$.

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Bibliography

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